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# Vibration of a Rotating Smart Beam

Shueei-Muh Lin,* I-Chun Mao,† and Juan-Huang Lin†
*Kun Shan University, Tainan 71003, Taiwan, Republic of China*

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**This paper investigates the active damping of the vibration of a rotating beam by using a pair of piezoelectric sensor and actuator layers. It uses Hamilton's principle to derive the governing differential equations and the boundary conditions for the coupled axial-bending vibration of a piezoelectric rotating beam. The derivative control law is used to apply the active damping of a rotating beam. An analytical method to determine the transient response of a piezoelectric rotating beam is proposed. The relation among the energy dissipation, the frequency, and the decay rate is revealed. The influences of the gain factor, the hub radius, the setting angle, and the rotating speed on the oscillating frequency and the decay rate are investigated.**

## Nomenclature

$A_i$ = cross-sectional area of the beam, $w_b \int_{z_{i,\text{bottom}}}^{z_{i,\text{upper}}} \mathrm{d}z$
$a$ = $a_a + a_h + a_s$
$a_i$ = dimensionless rotary inertia per unit length, $\rho_i I_i/(\rho_h A_h L^2)$
$B_i$ = $w_b \int_{z_{i,\text{bottom}}}^{z_{i,\text{upper}}} z\,\mathrm{d}z$
$b$ = $b_a + b_h + b_s$
$b_i$ = dimensionless bending rigidity, $E_i I_i/(E_h I_h)$
$c$ = elastic stiffness constant
$D$ = electric displacement
$d_h$ = dimensionless piezoelectric parameter, $\{\mu_{33}^{\varepsilon} A_a z_s^2 [h_s e_{31,s}/(\mu_{33}^{\varepsilon} L h_a)]^2/(\rho_h A_h L^2)\} k_d^2$
$d_q$ = dimensionless piezoelectric parameter, $[-e_{31,a} e_{31,s} B_a z_s h_s/(\mu_{33}^{\varepsilon} L^2 h_a \sqrt{\rho_h A_h E_h I_h})] k_d$
$E$ = Young's modulus of the beam
$E_p$ = electric field intensity
$EA$ = $E_1 A_1 + E_3 A_3$
$EB$ = $E_1 B_1 + E_3 B_3$
$EI$ = $E_1 I_1 + E_2 I_2 + E_3 I_3$
$e$ = piezoelectric constant
$H$ = $\{\mu_{33}^{\varepsilon}[h_s e_{31,s}/(\mu_{33}^{\varepsilon} L h_a)]^2\} k_d^2$
$I_i$ = area moment inertia of the beam, $w_b \int_{z_{i,\text{bottom}}}^{z_{i,\text{upper}}} z^2\,\mathrm{d}z$
$j$ = imaginary unit
$L$ = length of the blade
$m$ = $m_s + m_h + m_a$
$m_i$ = dimensionless mass per unit length, $\rho_i A_i/(\rho_h A_h)$
$N$ = centrifugal force
$n$ = dimensionless centrifugal force, $\alpha^2 \int_{\xi}^{1} m(r+\chi)\,\mathrm{d}\chi$
$Q$ = $[-e_{31,a} e_{31,s} h_s/(\mu_{33}^{\varepsilon} L^2 h_a)] k_d$
$R$ = radius of the root
$r$ = dimensionless radius of the hub, $R/L$
$t$ = time variable
$u, v, w$ = displacements in the $x$, $y$, and $z$ directions, respectively
$W$ = dimensionless lateral displacements in the $z$ direction, $w/L$
$w_b$ = width of the beam
$x, y, z$ = principal frame coordinates of the blade
$z_c$ = the neutral axis, $\sum_{i=a,h,s} E_i h_i \left[\left(\sum_{j=a}^{i} h_j\right) - \frac{1}{2} h_i\right] \Big/ \sum_{i=a,h,s} E_i h_i$
$\alpha$ = dimensionless rotational speed, $\Omega L^2 \sqrt{\rho_h A_h/(E_h I_h)}$
$\beta_i, \gamma_i$ = coefficients of the governing equation
$\varepsilon$ = strain
$\zeta$ = decay rate
$\theta$ = setting angle
$\xi$ = dimensionless distance to the root of the beam, $x/L$
$\mu$ = permittivity
$\rho$ = mass density per unit volume of the beam
$\sigma$ = stress
$\tau$ = dimensionless time, $t\sqrt{E_h I_h/(\rho_h A_h L^4)}$
$\Omega$ = rotational speed
$\omega$ = dimensionless frequency, $\varpi \sqrt{\rho_h A_h L^4/(E_h I_h)}$
$\varpi$ = frequency

*Subscripts*

a = actuator
h = host beam
s = sensor
1, 2, 3 = in the $x$, $y$, and $z$ directions, respectively

*Superscripts*

* = independent of *

## I. Introduction

ROTATING beams, which have importance in many practical applications such as turbine blades, helicopter rotor blades, airplane propellers, and robot manipulators, have been investigated for a long time. An interesting review of the subject can be found in the papers by Leissa [1], Ramamurti and Balasubramanian [2], Rao [3], and Lin [4]. Much attention has been focused on the undamped vibration problems. Lin et al. [5] and Lin and Lee [6] studied the passive damping of a rotating beam. No analytical solution for the vibration of the dynamic system has been presented.

Turcotte et al. [7] studied the vibration of a mistuned bladed-disk assembly using nonrotational structurally damped beams. The structural damping was introduced through a complex bending rigidity. Patel and Ganapathi [8] studied the free torsional vibration of nonrotating damped sandwich beams. Friswell and Lees [9] studied the free vibration of simply supported nonrotating damped beams. Lin et al. [5,6] investigated the vibration and instability of a rotating structurally and viscously damped beam with an elastically restrained root and root damping. The complex frequency relations among different systems were revealed. The instability of divergence, oscillating, and nonoscillating motions were predicted exactly, via the relations. The preceding literature investigated the passive damped vibration problems. Piezoelectric materials have been applied to the active control of structural vibrations and noises. Because of the complexity of analytical methods, the approximated

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*Professor, Mechanical Engineering Department.

†Graduate Student, Mechanical Engineering Department.

finite element method has been investigated by many researchers [10,11]. So far, little research has been done on the active-damping problem of a piezoelectric rotating beam, because of its complexity.

In this paper, the governing differential equations and the boundary conditions for the coupled axial-bending vibration of a piezoelectric rotating beam are derived by Hamilton's principle. An analytical method to determine the transient response of a piezoelectric rotating beam is derived. A system composed of the complex differential equations is written as a coupled real system expressed in terms of real and imaginary variables. A frequency equation is derived in terms of the eight normalized fundamental solutions of the two coupled characteristic governing differential equations. The relation among the energy dissipation, the frequency, and the decay rate is studied. The influences of the gain factor, the setting angle, and the rotating speed on the natural frequencies and the decay rate are investigated.

## II. Governing Equations and Boundary and Initial Conditions

Consider the transient response of a rotating smart beam mounted with $\theta$ on a hub with $R$, rotating with constant angular velocity $\Omega$. The upper and bottom surfaces of the beam are bonded piezoelectric sensor and actuator layers, as shown in Fig. 1. A set of differential equations of coupled axial-bending motion for the rotating beam are derived based on the following assumptions:

1) If the beam is assumed to be narrow in both the $y$ and $z$ directions and not loaded in these directions, then $\sigma_2 = \sigma_3 = 0$.

2) The shear deformation is negligible.

3) The rotary inertia is considered.

4) The transverse displacement is the same for all three layers.

5) The linear theory of piezoelectricity is applicable.

6) The electric field will be applied to the piezoelectric actuator on the $z$ direction (perpendicular to the planes of piezoelectric film). Therefore, $E_{p1} = E_{p2} = 0$.

The displacement fields of the beam are

$$u = u_0(x,t) - z\frac{\partial w(x,t)}{\partial x}, \qquad v = 0, \qquad w = w(x,t) \tag{1}$$

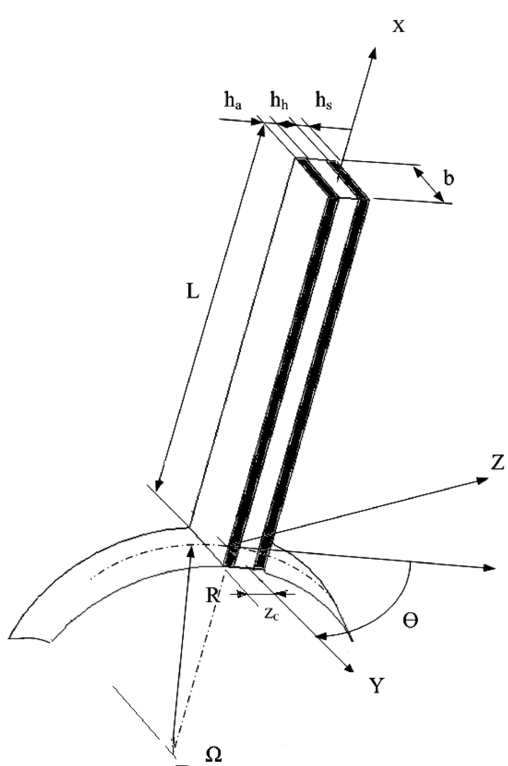


**Fig. 1 Geometry and coordinate system of a rotating beam.**

Based on the effect of centrifugal force, the nonlinear term of strain of the host beam is considered

$$\varepsilon_1 = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \varepsilon_2 = 0, \qquad \varepsilon_3 = 0 \tag{2}$$

The kinetic energy $T$ of the beam is

$$T = \frac{1}{2}\int_0^L \int_A \boldsymbol{V}\cdot\boldsymbol{V}\rho\,\mathrm{d}A\,\mathrm{d}x \tag{3}$$

where the velocity vector of a point $(x, y, z)$ in the beam is

$$\boldsymbol{V} = \left[\frac{\partial u}{\partial t} + \Omega\sin\theta(z+w) + y\Omega\cos\theta\right]i + [(x+R + u)\Omega\cos\theta]j + \left[\frac{\partial w}{\partial t} - (x+R+u)\Omega\sin\theta\right]k \tag{4}$$

The extended potential energy including the electric contribution is

$$U = \frac{1}{2}\int_\upsilon \sigma_1\varepsilon_1\,\mathrm{d}\upsilon - \frac{1}{2}\int_\upsilon E_{p3}D_3\,\mathrm{d}\upsilon \tag{5}$$

The constitutive equation of the piezoelectric material is

$$\sigma_1 = c_{11}^E\varepsilon_1 - e_{31}E_{p_3}, \qquad D_3 = e_{31}\varepsilon_1 + \mu_{33}^\varepsilon E_{p3} \tag{6}$$

The piezoelectric layer is used to sense the vibration of the rotating beam. The charge accumulated on the layer due to the direct piezoelectric effect is evaluated by

$$q = w_b\int e_{31}\varepsilon_1\,\mathrm{d}x \tag{7}$$

Considering the sensor to be a parallel capacitor, the voltage of the sensor is

$$V_s = \frac{h_s}{\mu_{33}^E L}\int e_{31}\varepsilon_1\,\mathrm{d}x \tag{8}$$

In closed-loop control, the control voltage on the piezoelectric actuator is designed by the following derivative control law [12]

$$V_a = -k_d\frac{\partial V_s}{\partial t} \tag{9}$$

where $k_d$ is a gain factor. Application of Hamilton's principle yields the following governing differential equations:

$$-\rho A\frac{\partial^2 u_0}{\partial t^2} + \rho B\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial t^2}\right) - 2\rho A\Omega\sin\theta\frac{\partial w}{\partial t} - \rho B\Omega^2\frac{\partial w}{\partial x} + \rho A\Omega^2(x + u_0 + R) + \frac{\partial N}{\partial x} + EA\frac{\partial^2 u_0}{\partial x^2} - EB\frac{\partial^3 w}{\partial x^3} - 2QA_1\frac{\partial^2 u_0}{\partial x\partial t} + Q(B_1 + A_1 z_3)\frac{\partial^3 w}{\partial t\partial x^2} - HA_1\frac{\partial^2 u_0}{\partial t^2} + HA_1 z_3\frac{\partial^3 w}{\partial t^2\partial x} = 0 \tag{10}$$

$$
\begin{aligned}
&-\rho B\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_0}{\partial x}\right)+\rho I\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2}\right)+\rho B\Omega^2\frac{\partial u_0}{\partial x}-\rho I\Omega^2\frac{\partial^2 w}{\partial x^2}\\
&\quad+2\rho A\Omega\sin\theta\frac{\partial u_0}{\partial t}-4\rho B\Omega\sin\theta\frac{\partial^2 w}{\partial t\partial x}-\rho A\frac{\partial^2 w}{\partial t^2}\\
&\quad+\rho B\Omega^2\sin^2\theta+\rho Aw\Omega^2\sin^2\theta+\frac{\partial}{\partial x}\left(N\frac{\partial w}{\partial x}\right)+EB\frac{\partial^3 u_0}{\partial x^3}\\
&\quad-EI\frac{\partial^4 w}{\partial x^4}-Q(B_1+A_1z_3)\frac{\partial^3 u_0}{\partial x^2\partial t}+2QB_1z_3\frac{\partial^4 w}{\partial t\partial x^3}\\
&\quad-HA_1z_3\frac{\partial^3 u_0}{\partial x\partial t^2}+HA_1z_3^2\frac{\partial^4 w}{\partial t^2\partial x^2}=0 \qquad (11)
\end{aligned}
$$

It should be noted that there exists a term in Eq. (11), $\rho B\Omega^2\sin^2\theta$, which is independent of time. It represents a small transverse static centrifugal force due to the coupled effect of the rotational speed and the small difference between the geometric center and the neutral one at which the bending stress is zero. Because it will not affect the dynamic behavior, the static term is negligible in studying the dynamic behavior of the beam.

The associated boundary conditions at $x=0$ are

$$u_0=0 \qquad (12)$$

$$\frac{\partial w}{\partial x}=0 \qquad (13)$$

$$w=0 \qquad (14)$$

and at $x=L$,

$$N+EA\frac{\partial u_0}{\partial x}-EB\frac{\partial^2 w}{\partial x^2}-QA_1\frac{\partial u_0}{\partial t}+QA_1z_3\frac{\partial^2 w}{\partial t\partial x}=0 \qquad (15)$$

$$EB\frac{\partial u_0}{\partial x}-EI\frac{\partial^2 w}{\partial x^2}-QB_1\frac{\partial u_0}{\partial t}+QB_1z_3\frac{\partial^2 w}{\partial t\partial x}=0 \qquad (16)$$

$$
\begin{aligned}
&\rho B\frac{\partial^2 u_0}{\partial t^2}-\rho I\frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial x}\right)+\rho I\Omega^2\frac{\partial w}{\partial x}-\rho B\Omega^2(x+u_0+R)\\
&\quad+2\rho B\Omega\sin\theta\frac{\partial w}{\partial t}-N\frac{\partial w}{\partial x}-EB\frac{\partial^2 u_0}{\partial x^2}+EI\frac{\partial^3 w}{\partial x^3}\\
&\quad+Q(B_1+A_1z_3)\frac{\partial^2 u_0}{\partial x\partial t}-2QB_1z_3\frac{\partial^3 w}{\partial t\partial x^2}+HA_1z_3\frac{\partial^2 u_0}{\partial t^2}\\
&\quad-HA_1z_3^2\frac{\partial^3 w}{\partial t^2\partial x}=0 \qquad (17)
\end{aligned}
$$

Assume that the beam is inextensional and the Coriolis force effect is neglected. Moreover, because the sensor and actuator layers are considered to be thin, the axial force in Eq. (10) is dominated by the centrifugal force $\rho A\Omega^2(x+R)$. As a result, the centrifugal force can be expressed as

$$N(x)=\Omega^2\int_x^L\rho(x)A(x)(R+x)\,\mathrm{d}x \qquad (18)$$

The governing equation in terms of the transverse displacement $w$ becomes

$$
\begin{aligned}
&\rho I\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2}\right)-\rho I\Omega^2\frac{\partial^2 w}{\partial x^2}-\rho A\frac{\partial^2 w}{\partial t^2}+\rho Aw\Omega^2\sin^2\theta\\
&\quad+\frac{\partial}{\partial x}\left(N\frac{\partial w}{\partial x}\right)-EI\frac{\partial^4 w}{\partial x^4}+2QB_1z_3\frac{\partial^4 w}{\partial t\partial x^3}+HA_1z_3^2\frac{\partial^4 w}{\partial t^2\partial x^2}=0 \qquad (19)
\end{aligned}
$$

The boundary conditions at $x=0$ are

$$\frac{\partial w}{\partial x}=0 \qquad (20)$$

$$w=0 \qquad (21)$$

and at $x=L$,

$$-EI\frac{\partial^2 w}{\partial x^2}+QB_1z_3\frac{\partial^2 w}{\partial t\partial x}=0 \qquad (22)$$

$$
\begin{aligned}
&-\rho I\frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial x}\right)+\rho I\Omega^2\frac{\partial w}{\partial x}-N\frac{\partial w}{\partial x}+EI\frac{\partial^3 w}{\partial x^3}-2QB_1z_3\frac{\partial^3 w}{\partial t\partial x^2}\\
&\quad-HA_1z_3^2\frac{\partial^3 w}{\partial t^2\partial x}=0 \qquad (23)
\end{aligned}
$$

It should be noted that when the sensor and the actuator are neglected, the differential equations are the same as those given by Lin [13].

In terms of the dimensionless quantities listed in the Nomenclature, the governing differential equation (19) and the boundary conditions (20–23) of the system are nondimensionalized as follows:

$$
\begin{aligned}
&a\frac{\partial^4 W}{\partial\xi^2\partial\tau^2}-a\alpha^2\frac{\partial^2 W}{\partial\xi^2}-m\frac{\partial^2 W}{\partial\tau^2}+mW\alpha^2\sin^2\theta+\frac{\partial}{\partial\xi}\left(n\frac{\partial W}{\partial\xi}\right)\\
&\quad-e\frac{\partial^4 W}{\partial\xi^4}+2d_q\frac{\partial^4 W}{\partial\xi^3\partial\tau}+d_h\frac{\partial^4 W}{\partial\xi^2\partial\tau^2}=0 \qquad (24)
\end{aligned}
$$

at $x=0$,

$$\frac{\partial W}{\partial\xi}=0 \qquad (25)$$

$$W=0 \qquad (26)$$

and at $x=1$,

$$-e\frac{\partial^2 W}{\partial\xi^2}+d_q\frac{\partial^2 W}{\partial\xi\partial\tau}=0 \qquad (27)$$

$$
\begin{aligned}
&-a\frac{\partial^3 W}{\partial\xi\partial\tau^2}+a\alpha^2\frac{\partial W}{\partial\xi}-n\frac{\partial W}{\partial\xi}+e\frac{\partial^3 W}{\partial\xi^3}-2d_q\frac{\partial^3 W}{\partial\xi^2\partial\tau}-d_h\frac{\partial^3 W}{\partial\xi\partial\tau^2}\\
&\quad=0 \qquad (28)
\end{aligned}
$$

The dimensionless initial conditions of the motion at the tip are

$$W(1,0)=w_0 \quad\text{and}\quad \frac{\partial W(1,0)}{\partial\tau}=\dot{w}_0 \qquad (29)$$

## III. Solution Method

### A. Characteristic Governing Equations and Boundary Conditions

Assume that the solution to Eqs. (24–29) is

$$W(\xi,\tau)=\tilde{W}(\xi)e^{\lambda\tau} \qquad (30a)$$

where $\tilde{W}$ represents the complex mode function and $\lambda$ is the complex frequency. They can be expressed as

$$\tilde{W}(\xi) = \bar{W}_R(\xi) + j\bar{W}_I(\xi), \qquad \lambda = -\zeta + j\omega \tag{30b}$$

The imaginary term $\omega$ is the damped frequency. Letting $ReW(1,0) = w_0$ and $Re\dot{W}(1,0) = \dot{w}_0$, one obtains

$$W_R(1) = w_0 \quad \text{and} \quad \bar{W}_I(1) = \frac{-1}{\omega}[\zeta w_0 + \dot{w}_0] \tag{31}$$

Substituting Eq. (30) into the governing equation (24) and the boundary conditions (25–28) and taking the real and imaginary parts apart, the coupled real differential equations can be obtained:

$$-e\frac{d^4\bar{W}_R}{d\xi^4} - 2d_q\left(\zeta\frac{d^3\bar{W}_R}{d\xi^3} + \omega\frac{d^3\bar{W}_I}{d\xi^3}\right) + a\left[(\zeta^2-\omega^2-\alpha^2)\frac{d^2\bar{W}_R}{d\xi^2} + 2\zeta\omega\frac{d^2\bar{W}_I}{d\xi^2}\right] + n\frac{d^2\bar{W}_R}{d\xi^2} + d_h\left[(\zeta^2-\omega^2)\frac{d^2\bar{W}_R}{d\xi^2} + 2\zeta\omega\frac{d^2\bar{W}_I}{d\xi^2}\right] + \frac{dn}{d\xi}\frac{d\bar{W}_R}{d\xi} - m[(\zeta^2-\omega^2-\alpha^2\sin^2\theta)\bar{W}_R + 2\zeta\omega\bar{W}_I] = 0 \tag{32a}$$

$$-e\frac{d^4\bar{W}_I}{d\xi^4} - 2d_q\left(\zeta\frac{d^3\bar{W}_I}{d\xi^3} - \omega\frac{d^3\bar{W}_R}{d\xi^3}\right) + a\left[(\zeta^2-\omega^2-\alpha^2)\frac{d^2\bar{W}_I}{d\xi^2} - 2\zeta\omega\frac{d^2\bar{W}_R}{d\xi^2}\right] + n\frac{d^2\bar{W}_I}{d\xi^2} + d_h\left[(\zeta^2-\omega^2)\frac{d^2\bar{W}_I}{d\xi^2} - 2\zeta\omega\frac{d^2\bar{W}_R}{d\xi^2}\right] + \frac{dn}{d\xi}\frac{d\bar{W}_I}{d\xi} - m[(\zeta^2-\omega^2-\alpha^2\sin^2\theta)\bar{W}_I - 2\zeta\omega\bar{W}_R] = 0 \tag{32b}$$

at $\xi = 0$,

$$\bar{W}_R = 0 \tag{33a}$$

$$\bar{W}_I = 0 \tag{33b}$$

$$\frac{d\bar{W}_R}{d\xi} = 0 \tag{34a}$$

$$\frac{d\bar{W}_I}{d\xi} = 0 \tag{34b}$$

at $\xi = 1$,

$$e\frac{d^2\bar{W}_R}{d\xi^2} + d_q\left(\zeta\frac{d\bar{W}_R}{d\xi} + \omega\frac{d\bar{W}_I}{d\xi}\right) = 0 \tag{35a}$$

$$e\frac{d^2\bar{W}_I}{d\xi^2} - d_q\left(\omega\frac{d\bar{W}_R}{d\xi} - \zeta\frac{d\bar{W}_I}{d\xi}\right) = 0 \tag{35b}$$

$$e\frac{d^3\bar{W}_R}{d\xi^3} + 2d_q\left(\zeta\frac{d^2\bar{W}_R}{d\xi^2} + \omega\frac{d^2\bar{W}_I}{d\xi^2}\right) - a\left[(\zeta^2-\omega^2-\alpha^2)\frac{d\bar{W}_R}{d\xi} + 2\zeta\omega\frac{d\bar{W}_I}{d\xi}\right] - d_h\left[(\zeta^2-\omega^2)\frac{d\bar{W}_R}{d\xi} + 2\zeta\omega\frac{d\bar{W}_I}{d\xi}\right] = 0 \tag{36a}$$

$$e\frac{d^3\bar{W}_R}{d\xi^3} + 2d_q\left(\zeta\frac{d^2\bar{W}_R}{d\xi^2} + \omega\frac{d^2\bar{W}_I}{d\xi^2}\right) - a\left[(\zeta^2-\omega^2-\alpha^2)\frac{d\bar{W}_R}{d\xi} + 2\zeta\omega\frac{d\bar{W}_I}{d\xi}\right] - d_h\left[(\zeta^2-\omega^2)\frac{d\bar{W}_R}{d\xi} + 2\zeta\omega\frac{d\bar{W}_I}{d\xi}\right] = 0 \tag{36b}$$

## B. Frequency Equations

The fundamental solution of the characteristic differential equations (32) is assumed to be

$$\begin{bmatrix} \bar{W}_R(\xi) \\ \bar{W}_I(\xi) \end{bmatrix} = \sum_{i=1}^{8} C_i \begin{bmatrix} \bar{W}_{R,i}(\xi) \\ \bar{W}_{I,i}(\xi) \end{bmatrix} \tag{37}$$

where the eight linearly independent fundamental solutions $[\bar{W}_{R,i}(\xi) \quad \bar{W}_{I,i}(\xi)]^T$, i = 1, 2, ..., 8, of Eq. (32) are chosen such that they satisfy the following normalization conditions at the origin of the coordinated system:

$$\begin{bmatrix} \bar{W}_{R,1} & \bar{W}_{R,2} & \bar{W}_{R,3} & \bar{W}_{R,4} & \bar{W}_{R,5} & \bar{W}_{R,6} & \bar{W}_{R,7} & \bar{W}_{R,8} \\ \bar{W}'_{R,1} & \bar{W}'_{R,2} & \bar{W}'_{R,3} & \bar{W}'_{R,4} & \bar{W}'_{R,5} & \bar{W}'_{R,6} & \bar{W}'_{R,7} & \bar{W}'_{R,8} \\ \bar{W}''_{R,1} & \bar{W}''_{R,2} & \bar{W}''_{R,3} & \bar{W}''_{R,4} & \bar{W}''_{R,5} & \bar{W}''_{R,6} & \bar{W}''_{R,7} & \bar{W}''_{R,8} \\ \bar{W}'''_{R,1} & \bar{W}'''_{R,2} & \bar{W}'''_{R,3} & \bar{W}'''_{R,4} & \bar{W}'''_{R,5} & \bar{W}'''_{R,6} & \bar{W}'''_{R,7} & \bar{W}'''_{R,8} \\ \bar{W}_{I,1} & \bar{W}_{I,2} & \bar{W}_{I,3} & \bar{W}_{I,4} & \bar{W}_{I,5} & \bar{W}_{I,6} & \bar{W}_{I,7} & \bar{W}_{I,8} \\ \bar{W}'_{I,1} & \bar{W}'_{I,2} & \bar{W}'_{I,3} & \bar{W}'_{I,4} & \bar{W}'_{I,5} & \bar{W}'_{I,6} & \bar{W}'_{I,7} & \bar{W}'_{I,8} \\ \bar{W}''_{I,1} & \bar{W}''_{I,2} & \bar{W}''_{I,3} & \bar{W}''_{I,4} & \bar{W}''_{I,5} & \bar{W}''_{I,6} & \bar{W}''_{I,7} & \bar{W}''_{I,8} \\ \bar{W}'''_{I,1} & \bar{W}'''_{I,2} & \bar{W}'''_{I,3} & \bar{W}'''_{I,4} & \bar{W}'''_{I,5} & \bar{W}'''_{I,6} & \bar{W}'''_{I,7} & \bar{W}'''_{I,8} \end{bmatrix}_{\xi=0} = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix} \tag{38}$$

where primes indicate differentiation with respect to the dimensionless spatial variable $\xi$.

Substituting Eq. (37) into the boundary conditions (33) and (34), one obtains the coefficients $C_1 = C_2 = C_5 = C_6 = 0$. Further, substituting the solution into Eq. (35) and the initial conditions (31), the following relation is obtained:

$$\begin{bmatrix} \kappa_3 & \kappa_4 & \kappa_7 & \kappa_8 \\ \bar{\kappa}_3 & \bar{\kappa}_4 & \bar{\kappa}_7 & \bar{\kappa}_8 \\ \bar{W}_{R,3}(1) & \bar{W}_{R,4}(1) & \bar{W}_{R,7}(1) & \bar{W}_{R,8}(1) \\ \bar{W}_{I,3}(1) & \bar{W}_{I,4}(1) & \bar{W}_{I,7}(1) & \bar{W}_{I,8}(1) \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \\ C_7 \\ C_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ w_0 \\ \frac{-1}{\omega}[\zeta w_0 + \dot{w}_0] \end{bmatrix} \tag{39a}$$

where

$$\begin{aligned} \kappa_i &= eW''_{R,i}(1) + d_q[\zeta W'_{R,i}(1) + \omega W'_{I,i}(1)] \\ \bar{\kappa}_i &= eW''_{I,i}(1) + d_q[\zeta W'_{I,i}(1) + \omega W'_{R,i}(1)] \end{aligned} \tag{39b}$$

Given the initial displacement $w_0$ and velocity $\dot{w}_0$, the oscillating frequency and the decay rate can be easily determined via the coupled frequency Eq. (36) by using the numerical method proposed by Lin [13].

### C. Exact Fundamental Solutions

In general, the closed-form fundamental solutions of two coupled differential equations with variable coefficients cannot be determined. However, if the coefficients of the equations, which involve the material properties and/or geometric parameters, can be expressed in matrix polynomial form, then a power series representation of the fundamental solutions can be constructed by the modified method of Frobinius [6]. Equation (32) can be expressed as

$$\bar{A}_1\frac{d^4\bar{W}_R}{d\xi^4}+\bar{A}_2\frac{d^3\bar{W}_R}{d\xi^3}+\bar{A}_3\frac{d^2\bar{W}_R}{d\xi^2}+\bar{A}_4\frac{d\bar{W}_R}{d\xi}+\bar{A}_5\bar{W}_R+\bar{A}_6\frac{d^3\bar{W}_I}{d\xi^3}+\bar{A}_7\frac{d^2\bar{W}_I}{d\xi^2}+\bar{A}_8\bar{W}_I=0 \tag{40a}$$

$$\tilde{A}_1\frac{d^4\bar{W}_I}{d\xi^4}+\tilde{A}_2\frac{d^3\bar{W}_I}{d\xi^3}+\tilde{A}_3\frac{d^2\bar{W}_I}{d\xi^2}+\tilde{A}_4\frac{d\bar{W}_I}{d\xi}+\tilde{A}_5\bar{W}_I+\tilde{A}_6\frac{d^3\bar{W}_R}{d\xi^3}+\tilde{A}_7\frac{d^2\bar{W}_R}{d\xi^2}+\tilde{A}_8\bar{W}_R=0,\qquad \xi\in(0,1) \tag{40b}$$

where

$$\begin{gathered}\bar{A}_1=a_0,\quad \tilde{A}_1=\bar{a}_0,\quad \bar{A}_2=b_0,\quad \tilde{A}_2=\bar{b}_0\\ \bar{A}_3=c_0+c_1\xi+c_2\xi^2,\quad \tilde{A}_3=\bar{c}_0+\bar{c}_1\xi+\bar{c}_2\xi^2\\ \bar{A}_4=d_0+d_1\xi,\quad \tilde{A}_4=\bar{d}_0+\bar{d}_1\xi,\quad \bar{A}_5=e_0,\quad \tilde{A}_5=\bar{e}_0\\ \bar{A}_6=f_0,\quad \tilde{A}_6=\bar{f}_0,\quad \bar{A}_7=g_0,\quad \tilde{A}_7=\bar{g}_0,\quad \bar{A}_8=h_0\\ \tilde{A}_8=\bar{h}_0\end{gathered} \tag{41a}$$

in which

$$\begin{gathered}a_0=\bar{a}_0=-e,\qquad b_0=\bar{b}_0=-2D_q\zeta\\ c_0=\bar{c}_0=a(\zeta^2-\omega^2-\alpha^2)+D_h(\zeta^2-\omega^2)+\alpha^2m(r+1/2)\\ c_1=\bar{c}_1=-r\alpha^2m,\qquad c_2=\bar{c}_2=-\frac{1}{2}\alpha^2m\\ d_0=\bar{d}_0=-r\alpha^2m,\qquad d_1=\bar{d}_1=-\alpha^2m\\ e_0=\bar{e}_0=-m(\zeta^2-\omega^2-\alpha^2\sin^2\theta),\qquad f_0=-\bar{f}_0=-2D_q\omega\\ g_0=-\bar{g}_0=2\zeta\omega a+2\zeta\omega D_h,\qquad h_0=-\bar{h}_0=-2\zeta\omega m\end{gathered} \tag{41b}$$

One can assume that the eight fundamental solutions of Eq. (32) are in the form of

$$\begin{bmatrix}\bar{W}_{R,j}\\ \bar{W}_{I,j}\end{bmatrix}=\sum_{k=0}^{\infty}\begin{bmatrix}\alpha_{j,k}\xi^k\\ \beta_{j,k}\xi^k\end{bmatrix},\qquad \mathrm{j}=1,2,\ldots,8 \tag{42}$$

and

$$\begin{gathered}\text{for }\bar{\mathrm{W}}_{R,1}\text{: }\alpha_{1,0}=1,\qquad \alpha_{1,1}=\alpha_{1,2}=\alpha_{1,3}=0\\ \text{for }\bar{\mathrm{W}}_{R,2}\text{: }\alpha_{2,1}=1,\qquad \alpha_{2,0}=\alpha_{12}=\alpha_{2,3}=0\\ \text{for }\bar{\mathrm{W}}_{R,3}\text{: }\alpha_{3,2}=1/2,\qquad \alpha_{3,0}=\alpha_{3,1}=\alpha_{3,3}=0\\ \text{for }\bar{\mathrm{W}}_{R,4}\text{: }\alpha_{4,3}=1/6,\qquad \alpha_{1,1}=\alpha_{1,2}=\alpha_{1,3}=0\\ \text{for }\bar{\mathrm{W}}_{R,5}\text{: }\alpha_{5,0}=\alpha_{5,1}=\alpha_{5,2}=\alpha_{5,3}=0\\ \text{for }\bar{\mathrm{W}}_{R,6}\text{: }\alpha_{6,0}=\alpha_{6,1}=\alpha_{6,2}=\alpha_{6,3}=0\\ \text{for }\bar{\mathrm{W}}_{R,7}\text{: }\alpha_{7,0}=\alpha_{7,1}=\alpha_{7,2}=\alpha_{7,3}=0\\ \text{for }\bar{\mathrm{W}}_{R,8}\text{: }\alpha_{8,0}=\alpha_{8,1}=\alpha_{8,2}=\alpha_{8,3}=0\\ \text{for }\bar{\mathrm{W}}_{I,1}\text{: }\beta_{1,0}=\beta_{1,1}=\beta_{1,2}=\beta_{1,3}=0\\ \text{for }\bar{\mathrm{W}}_{I,2}\text{: }\beta_{2,0}=\beta_{2,1}=\beta_{2,2}=\beta_{2,3}=0\\ \text{for }\bar{\mathrm{W}}_{I,3}\text{: }\beta_{3,0}=\beta_{3,1}=\beta_{3,2}=\beta_{3,3}=0\\ \text{for }\bar{\mathrm{W}}_{I,4}\text{: }\beta_{4,0}=\beta_{4,1}=\beta_{4,2}=\beta_{4,3}=0,\qquad \text{for }\bar{\mathrm{W}}_{I,5}\text{: }\beta_{5,0}=1\\ \beta_{5,1}=\beta_{5,2}=\beta_{5,3}=0,\qquad \text{for }\bar{\mathrm{W}}_{I,6}\text{: }\beta_{6,1}=1\\ \beta_{6,0}=\beta_{6,2}=\beta_{6,3}=0,\qquad \text{for }\bar{\mathrm{W}}_{I,7}\text{: }\beta_{7,2}=1/2\\ \beta_{7,0}=\beta_{7,1}=\beta_{7,3}=0,\qquad \text{for }\bar{\mathrm{W}}_{I,8}\text{: }\beta_{8,3}=1/6\\ \beta_{8,0}=\beta_{8,1}=\beta_{8,2}=0\end{gathered} \tag{43}$$

These eight fundamental solutions satisfy the normalization condition (38). Upon substituting Eq. (42) into Eq. (32) and collecting the coefficients of like powers of $\xi$, the following recurrence formulas can be obtained:

$$\begin{aligned}\alpha_{j,4}&=\frac{-1}{24a_0}(h_0\beta_{j,0}+2g_0\beta_{j,2}+6f_0\beta_{j,3}+e_0\alpha_{j,0}+d_0\alpha_{j,1}\\&\quad+2c_0\alpha_{j,2}+6b_0\alpha_{j,3})\\ \beta_{j,4}&=\frac{-1}{24\bar{a}_0}(\bar{h}_0\alpha_{j,0}+2\bar{g}_0\alpha_{j,2}+6\bar{f}_0\alpha_{j,3}+\bar{e}_0\beta_{j,0}+\bar{d}_0\beta_{j,1}\\&\quad+2\bar{c}_0\beta_{j,2}+6\bar{b}_0\beta_{j,3})\\ \alpha_{j,m+4}&=\frac{-1}{a_0(m+4)(m+3)(m+2)(m+1)}\Big[h_0\beta_{j,m}+g_0(m\\&\quad+2)(m+1)\beta_{j,m+2}+f_0(m+3)(m+2)(m+1)\beta_{j,m+3}\\&\quad+e_0\alpha_{j,m}+\sum_{k=0}^{m}\mathrm{d}_k(\mathrm{m}-\mathrm{k}+1)\alpha_{j,m-k+1}+\sum_{k=0}^{m}\mathrm{c}_k(\mathrm{m}-\mathrm{k}+2)(\mathrm{m}\\&\quad-\mathrm{k}+1)\alpha_{j,m-k+2}+b_0(m+3)(m+2)(m+1)\alpha_{j,m+3}\Big]\\ \beta_{j,m+4}&=\frac{-1}{\bar{a}_0(m+4)(m+3)(m+2)(m+1)}\Big[\bar{h}_0\alpha_{j,m}+\bar{g}_0(m\\&\quad+2)(m+1)\alpha_{j,m+2}+\bar{f}_0(m+3)(m+2)(m+1)\alpha_{j,m+3}\\&\quad+\bar{e}_0\beta_{j,m}+\sum_{k=0}^{m}\bar{\mathrm{d}}_k(\mathrm{m}-\mathrm{k}+1)\beta_{j,m-k+1}+\sum_{k=0}^{m}\bar{\mathrm{c}}_k(\mathrm{m}-\mathrm{k}+2)(\mathrm{m}\\&\quad-\mathrm{k}+1)\beta_{j,m-k+2}+\bar{b}_0(m+3)(m+2)(m+1)\beta_{j,m+3}\Big]\\ m&=1,2,\ldots\end{aligned} \tag{44}$$

With these recurrence formulas (44), one can generate the eight exact normalized fundamental solutions of the differential equation (32). Consequently, upon substituting these fundamental solutions into the frequency equation (36), the exact complex frequencies of the beam are obtained.

## IV. Relation Among Energy Dissipation, Frequency Shift, and Decay Rate

Consider the transient behavior of a beam and the definition of the $Q$ factor. With the active damping of piezoelectricity, the amplitude of vibration decreases from the initial value. The total energy is a function of time. Neglecting the electrostatic energy of two piezoelectric thin layers, the strain energy of the beam is

$$E_s(\tau) = \frac{1}{2}\sum_{i=a,h,s} b_i \int_0^1 [ReW''(\xi,\tau)]^2 \mathrm{d}\xi \tag{45}$$

If the initial velocity is zero, the initial total energy is $E_t(0) = E_s(0)$. Similarly, the total energy through a period $T$ is $E_t(T) = E_s(T)$. The total energy and the energy lost through a cycle, $E_{\mathrm{loss}} = E_t(0) - E_t(T)$, are derived as

$$E_t(0) = \frac{1}{2}\sum_{i=a,h,s} b_i \int_0^1 [W_R''(\xi)]^2\,\mathrm{d}\xi$$
$$E_{\mathrm{loss}} = (1 - e^{-4\pi\zeta/\omega})\frac{1}{2}\sum_{i=a,h,s} b_i \int_0^1 [W_R''(\xi)]^2\,\mathrm{d}\xi \tag{46}$$

Substituting Eq. (46) into the definition of the $Q$ factor, one obtains

$$Q\ \text{factor} = 2\pi\frac{E_t}{E_{\mathrm{loss}}} = \frac{2\pi}{(1 - e^{-4\pi\zeta/\omega})} \tag{47}$$

## V. Numerical Results and Discussion

To demonstrate the efficiency and convergence of the proposed method to solve the vibration problem, the transient response, the frequency shift, and the decay rate of a rotating beam are determined. Figure 2 demonstrates the transient response of the beam. The initial displacement and velocity are given. The larger the gain factor, the faster the decay of the oscillation of the beam. In Table 1, the convergence pattern of the complex eigenvalues of the beam is shown. It shows that the eigenvalues determined by the proposed method converge very rapidly. The convergent frequency shift without the piezoelectric damping is the same as that given by Lin [13].

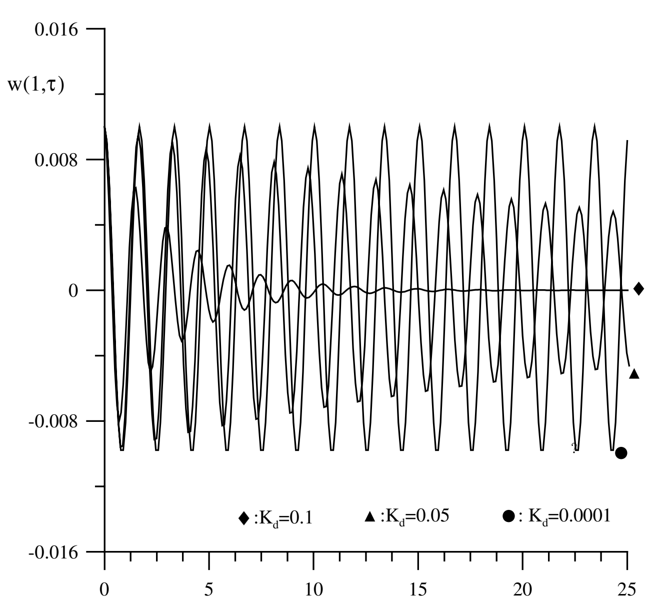


**Fig. 2 Oscillation of a rotating cantilever beam [$w_0 = 0.01$, $\dot{w} = 0$, $a = 0.00001$, $e = m = 1$, $d_h = 0.98k_d^2$, $d_q = 1.892k_d$, $r = 0.5$, $\alpha = 1$, and $\theta = 30\,\mathrm{deg}$].**

**Table 1 Convergence pattern of the first eigenvalue of a rotating cantilever beam [$w_0 = 0.01$, $\dot{w} = 0$, $a = 0.00001$, $e = m = 1$, $d_h = 0.98k_d^2$, $d_q = 1.892k_d$, $r = 0.5$, $\alpha = 1$, and $\theta = 30\,\mathrm{deg}$]**

| No. of terms | $k_d = 0$ | $k_d = 0.05$ | | $k_d = -0.03$ | |
|---|---|---|---|---|---|
| | $\omega$ | $\omega$ | $\zeta$ | $\omega$ | $\zeta$ |
| 10 | 3.7562 | 3.6709 | 0.004021 | 3.7253 | −0.000805 |
| 20 | 3.7534 | 3.6652 | 0.011003 | 3.7211 | −0.002479 |
| 30 | 3.7534 | 3.6652 | 0.011003 | 3.7211 | −0.002479 |
| 40 | 3.7534 | 3.6652 | 0.011003 | 3.7211 | −0.002479 |
| 50 | 3.7534 | 3.6652 | 0.011003 | 3.7211 | −0.002479 |
| [13] | 3.7534 | ~ | ~ | ~ | ~ |

Figure 3 shows the influence of the gain factor and the rotational speed on the frequency and the decay rate. It is found that when the gain factor is increased or decreased from zero, the fundamental frequency is decreased. A positive or negative gain factor results in a corresponding positive or negative decay rate. It means that if the decay rate is positive, the amplitude decays exponentially and the system is stable; otherwise, it is unstable. Increasing the gain factor from zero, the decay rate increases rapidly from zero to a critical value and then decreases slowly. The influence of the gain factor on

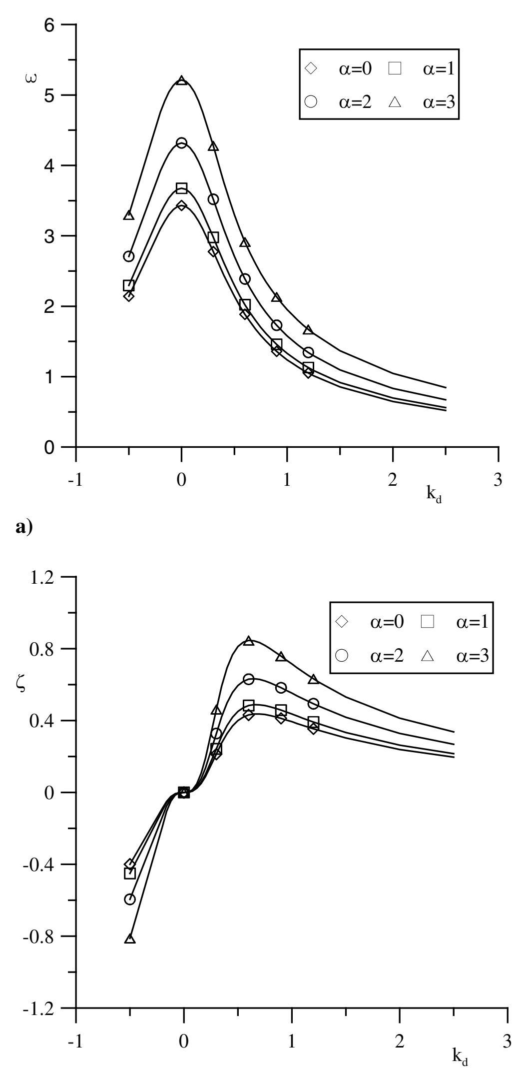


**Fig. 3 Influence of the gain factor $k_d$ and the rotational speed $\alpha$ on the frequency and decay rate of a rotating cantilever beam [$w_0 = 0.01$, $\dot{w} = 0$, $a = 0.0000096$, $e = 1.722$, $m = 1.807$, $d_h = 0.98k_d^2$, $d_q = 1.892k_d$, $r = 0.5$, and $\theta = 30\,\mathrm{deg}$].**

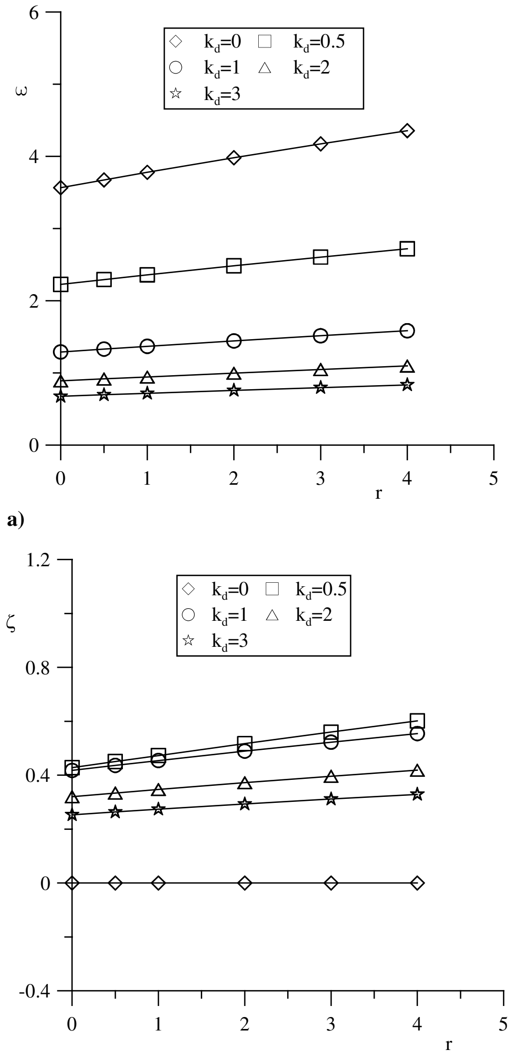


**Fig. 4 Influence of the gain factor $k_d$ and the radius of hub $r$ on the frequency and the decay rate of a rotating cantilever beam [$w_0 = 0.01$, $\dot{w} = 0$, $a = 0.0000096$, $e = 1.722$, $m = 1.807$, $d_h = 0.98k_d^2$, $d_q = 1.892k_d$, $\alpha = 1$, and $\theta = 30\,\mathrm{deg}$].**

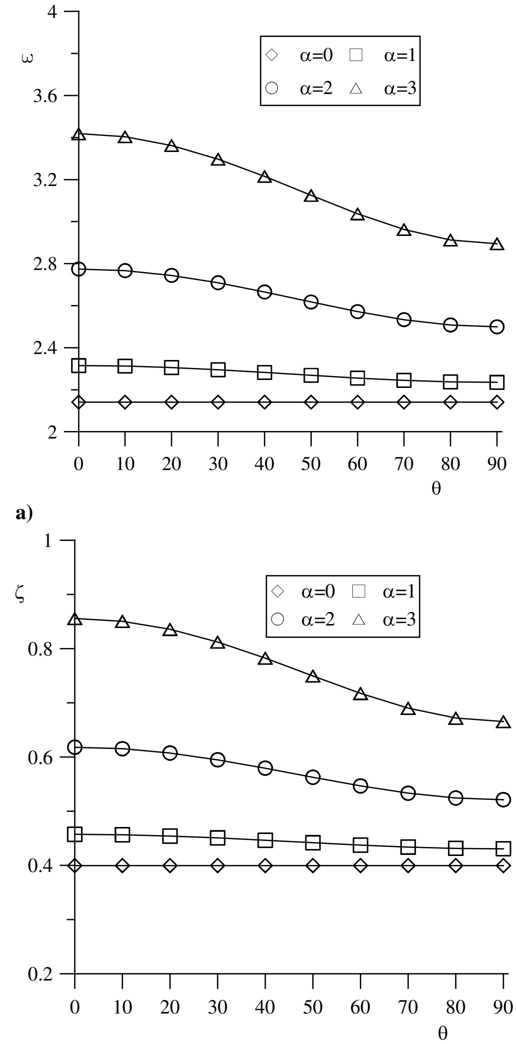


**Fig. 5 Influence of the setting angle and the rotational speed $\alpha$ on the frequency and the decay rate of a rotating cantilever beam [$w_0 = 0.01$, $\dot{w} = 0$, $a = 0.0000096$, $e = 1.722$, $m = 1.807$, $d_h = 0.98k_d^2$, $d_q = 1.892k_d$, $k_d = 0.5$, and $\alpha = 0.5$].**

the frequency and the decay rate is more obvious, especially at a larger rotational speed.

Figure 4 shows the influence of the gain factor $k_d$ and the dimensionless radius of hub $r$ on the frequency and the decay rate. It is well known that increasing the radius of hub $r$ increases the stiffness and the natural frequency of a rotating conventional beam. Figure 4 shows that increasing the hub radius increases the frequency and the decay rate. Moreover, increasing the gain factor from zero will increase the decay rate. However, when the gain factor is larger than a critical value, increasing the gain factor will decrease the decay rate. This phenomenon has also been verified in Fig. 3b. Figure 5 shows the influence of the setting angle and the rotational speed on the frequency and the decay rate, with $k_d = 0.5$. Increasing the setting angle decreases the frequency and the decay rate, especially at a larger rotational speed.

Figure 6 shows the influence of the gain factor $k_d$ on the $Q$ factor via Eq. (47). When $k_d$ is very small, the decay rate approaches zero and the $Q$ factor is very large. It means that energy dissipation is very small. When $k_d$ increases over a critical value, the $Q$ factor approaches a fixed value. It is easily understood via Eq. (47) and Fig. 3. Moreover, it is found that the influence of rotating speed on the $Q$ factor is slight. The $Q$ factor depends predominantly on the gain factor.

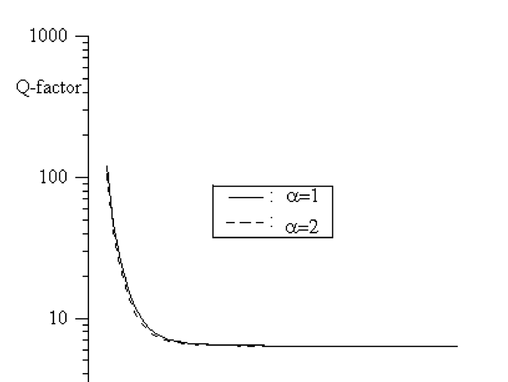


**Fig. 6 Influence of the gain factor $k_d$ on the $Q$ factor of a rotating cantilever beam [$w_0 = 0.01$, $\dot{w} = 0$, $a = 0.0000096$, $e = 1.722$, $m = 1.807$, $d_h = 0.98k_d^2$, $d_q = 1.892k_d$, $r = 0.5$, and $\theta = 30\,\mathrm{deg}$].**

## VI. Conclusions

In this paper, Hamilton's principle is used to derive the governing differential equations and the boundary conditions for the coupled axial-bending vibration of a piezoelectric rotating beam. The analytical solution of a piezoelectric rotating beam is derived. The relation among the energy dissipation, the frequency, and the decay rate is revealed. The influences of the gain factor, the setting angle, and the rotating speed on the frequency and the decay rate are investigated. The findings are as follows:

1) When the gain factor is increased or decreased from zero, the frequency is decreased. When the gain factor increases from zero, the decay rate increases rapidly from zero to a critical value and then decreases slowly.

2) Given a gain factor, increasing the setting angle decreases the frequency and the decay rate, especially at a larger rotational speed.

3) When the gain factor is large enough, the $Q$ factor is almost constant. The influence of rotating speed on the $Q$ factor is slight. The $Q$ factor depends predominantly on the gain factor.

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M. Ahmadian
*Associate Editor*